Characterizing the inscribability of polytopes using slack matrices

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The problem

Given a polytope, does there exist a combinatorially equivalent* polytope with all vertices on the sphere?

*The face lattices of the two polytopes are isomorphic

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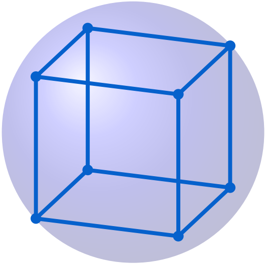
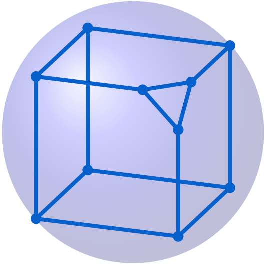
Given a polytope, does there exist a combinatorially equivalent* polytope with all vertices on the sphere?

Yes! ⇒ Inscribable



No! ⇒ Non-inscribable



*The face lattices of the two polytopes are isomorphic

Solving a nonlinear system

Theorem. [Firsching, 2017] A d-polytope P with n vertices is inscribable if and only if the following system has a solution

$$\chi(i_1, \ldots, i_r) \det(\overline{p}_{i_1}, \ldots, \overline{p}_{i_r}) > 0 \quad \text{for all } i_1, \ldots, i_r \in \binom{[n]}{r}$$
$$\|p_i\|^2 = 1 \quad \text{for } i = 1, \ldots, n$$

where $r = d + 1$, $\overline{p}_i = \begin{bmatrix} p_i \\ 1 \end{bmatrix}$, $[n] = \{1, \ldots, n\}$, and $\chi : [n]^r \to \{-1, 0, 1\}$ is the associated chirotope of P

Note: The system has nd variables, $\binom{n}{r}$ inequalities of degree r, and n equalities of degree 2

Slack matrix

Suppose P is a d-polytope such that

- $P = \mathrm{conv}\{v_1, \ldots, v_n\}$
- $P = \{x \in \mathbb{R}^d : 1 - h_j^\top x \geq 0, j = 1, \ldots, m\}$

Let $V = [v_1 \cdots v_n] \in \mathbb{R}^{d \times n}$ and $H = [h_1 \cdots h_m] \in \mathbb{R}^{d \times m}$

A slack matrix $S \in \mathbb{R}^{n \times m}$ of P is

$$S = \begin{bmatrix} \mathbb{1}_n & V^\top \end{bmatrix} \begin{bmatrix} \mathbb{1}_m^\top \\ -H \end{bmatrix} = \mathbb{1}_{n \times m} - V^\top H$$

Note: $S_{ij} = 1 - h_j^\top v_i$ is facet inequality j evaluated on vertex i

Characterizing inscribability using slack matrices

Theorem. A d-polytope P is inscribable if and only if there exists

$$X = \begin{bmatrix} 1 & \mathbb{1}_n^\top & \mathbb{1}_m^\top \\ \mathbb{1}_n & A & S \\ \mathbb{1}_m & S^\top & B \end{bmatrix} \succcurlyeq 0$$

such that

- $\mathrm{rank}(X) = d + 1$
- $\mathrm{diag}(A) = \mathrm{const.}$
- $S \geq 0$
- S has the same support as slack matrices of P

Characterizing inscribability using slack matrices

Let I^z be the index set of zeros in slack matrices
Consider the following optimization problem

$$\begin{aligned}
\min_X \quad & \operatorname{rank}(X) \\
s.t. \quad & X = \begin{bmatrix} 1 & \mathbb{1}_n^\top & \mathbb{1}_m^\top \\ \mathbb{1}_n & A & S \\ \mathbb{1}_m & S^\top & B \end{bmatrix} \succcurlyeq 0 \\
& S_{ij} = 0, \text{ if } (i,j) \in I^z \\
& S_{ij} > 0, \text{ if } (i,j) \notin I^z \\
& A_{ii} = 2, i = 1, \ldots, n.
\end{aligned}$$

Note: The minimum of this problem is no less than $d + 1$

Question: How to solve this?

An SDP formulation

An SDP formulation

Instead of solving the min-rank problem directly, we consider the following SDP problem

$$\begin{aligned}
\min_{X} \quad & \operatorname{tr}(X) - \sum_{(i,j)\notin I^z} \lambda_{ij} S_{ij} \\
s.t. \quad & X = \begin{bmatrix} 1 & \mathbb{1}_n^\top & \mathbb{1}_m^\top \\ \mathbb{1}_n & A & S \\ \mathbb{1}_m & S^\top & B \end{bmatrix} \succcurlyeq 0 \\
& S_{ij} = 0, \text{ if } (i,j) \in I^z \\
& A_{ii} = 2, i = 1, \dots, n
\end{aligned} \tag{P}$$

where λ_{ij} are some positive weights

Dual problem and duality gap

The dual problem of (P) is

$$\begin{aligned} \max_{u,v} \quad & m + n + \sum_{\substack{1\le i\le n\\ 1\le j\le m}} M_{ij} - \sum_{i=1}^{n} u_i + 1 \\ s.t. \quad & \begin{bmatrix} I_n + \mathrm{diag}(u) & \frac{1}{2}M \\ \frac{1}{2}M^\top & I_m \end{bmatrix} \succcurlyeq 0 \end{aligned} \tag{D}$$

where

$$\mathrm{diag}(u) = \begin{bmatrix} u_1 & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & u_n \end{bmatrix} \quad \text{and} \quad M_{ij} = \begin{cases} -\lambda_{ij}, & \text{if } (i,j) \notin I^z \\ v_k \text{ for some } k, & \text{if } (i,j) \in I^z \end{cases}$$

Note: The Slater's condition holds so the duality gap is zero

Works for inscribable polytopes?

For an inscribable polytope P

- Find an inscription of P and a set of weights λ_{ij} such that the duality gap is zero evaluated at the inscription
- Then, solving the SDP with the λ_{ij} above will give an inscription of P

Example: n-gons

For n-gons, we have $n = m \geq 3$ and $d = 2$
Problem (P) evaluated at the inscription

$$v_i = \left[\cos\frac{2(i-1)\pi}{n} \;\; \sin\frac{2(i-1)\pi}{n}\right]^\top, i = 1, \ldots, n$$

$$h_j = \frac{1}{\cos\frac{\pi}{n}}\left[\cos\frac{(2j-1)\pi}{n} \;\; \sin\frac{(2j-1)\pi}{n}\right]^\top, j = 1, \ldots, n$$

with

$$\lambda_{ij} = \frac{2}{n\cos^2\frac{\pi}{n}}, \forall (i,j) \notin I^z$$

has

$$f_p^* = f_d^* = 2n - n\tan^2\frac{\pi}{n} + 1$$

Therefore, the SDP formulation solves the inscribability problem for n-gons

More examples

For d-simplices, d-cubes, and d-crosspolytopes, the SDP formulation solves the inscribability problem

In particular, solving (P) with the following weights gives an inscription

$$\begin{aligned}
&(d\text{-simplex}) \quad && \lambda_{ij} = \frac{2d^2}{d+1}, && \forall (i,j) \notin I^z \\
&(d\text{-cube}) \quad && \lambda_{ij} = d2^{1-d}, && \forall (i,j) \notin I^z \\
&(d\text{-crosspolytope}) \quad && \lambda_{ij} = 1, && \forall (i,j) \notin I^z
\end{aligned}$$

Algorithms

Algorithms for the min-rank problem

- Solving the SDP formulation
- Alternating projection: project between the rank $d+1$ cone and the feasible set Ω of the min-rank problem
- Simplified alternating projection: replace the projection onto Ω with forcing X_k to have correct constants on correct positions

Numerical experiments

- Test set: inscribable simplicial 4-polytopes with $5 \leq n \leq 10$ vertices
- Starting point for AP and simplified AP is the solution of the SDP
- Results comparing accuracy:

Alg\\n	5	6	7	8	9	10
SDP	100%	100%	80%	84%	87%	59%
AP (simplified)	100%	100%	80%	86%	89%	61%
AP	100%	100%	100%	95%	97%	94%

- Results comparing runtime (s):

Alg\\n	5	6	7	8	9	10
SDP	0.4	0.5	0.6	0.9	1.3	1.5
AP (simplified)	0.4	0.5	1.1	1.5	2.7	7.16
AP	0.4	0.5	15.1	86.7	122.0	330.2

Summary

In this research, we

- Characterized the inscribability problem of polytopes as a min-rank optimization problem based on slack matrices
- Proposed an SDP formulation to approximate the min-rank problem and proved it works for certain classes of polytopes
- Provided three algorithms with different accuracy and efficiency for the inscribability problem

Thank you